ISRAEL JOURNAL OF MATHEMATICS 117 (2000), 71–92

STRONGLY BAIRE TREES AND A COFINAL BRANCH PRINCIPLE

BY

QI FENG*

Institute of Mathematics, Academia Sinica, Beijing, China 100080
e-mail: qifeng@math03.math.ac.cn, feng@logic.math.ac.cn

ABSTRACT

We study strongly Baire trees. The Cofinal Branch Principle is the statement that every strongly Baire tree of height ω_1 has a cofinal branch. We show that this principle implies that the strong reflection principle holds, there are no Souslin trees and $MA^+(\sigma\text{-closed})$ holds. Also it follows from the Semiproper Forcing Axiom, and the Strong Reflection Principle does not imply the cofinal branch principle.

1. Introduction

In [7], Foreman–Magidor–Shelah proved that Martin's Maximum is consistent relative to the existence of a supercompact cardinal. They proved that Martin's Maximum implies the nonstationary ideal NS_{ω_1} is saturated, the Singular Cardinal Hypothesis holds, together with some other things. Shelah then showed that Martin's Maximum is equivalent to the Semiproper Forcing Axiom in [15]. Todorcevic in [18, 19] (see [3]) formulated a Strong Reflection Principle and showed that the consequences of Martin's Maximum in [7] follow from this Strong Reflection Principle. Velickovic in [21] showed that a much weaker reflection principle implies the Singular Cardinal Hypothesis holds. In [5, 6], Feng–Jech studied the reflection principles and showed that Todorcevic's Strong Reflection Principle is

* Supported by a grant from Academia Sinica and a grant from The Chinese National Science Foundation (19871803).
Received June 13, 1998

equivalent to that every projective stationary set contains an increasing continuous $\in$-chain of length ω_1. In this paper, we continue these works to study the fine combinatorial consequences of the Semiproper Forcing Axiom. Our focusing will be on (ω, ∞)-distributive partially ordered sets, Baire trees in particular. We will show that if we assume every stronly Baire tree of height ω_1 has a cofinal branch, then the Strong Reflection Principle and $MA^+(\sigma\text{-closed})$ hold and there is no Souslin tree.

Let us first recall some basic terms from Jech [11, 13]. All undefined terms in this paper are taken from Jech [11, 13]. All the partially ordered sets are assumed to be separative. Namely, if $(P, \leq)$ is a partially ordered set, p, q are in P, and $p \not\leq q$, then there is $r \leq p$ such that r and q are incompatible (i.e., there is no $r_1 \leq r$ such that $r_1 \leq q$). Let $(P, \leq)$ be a partially ordered set. $D \subseteq P$ is dense in P if for every $p \in P$ there is $q \in D$ such that $q \leq p$. $D \subseteq P$ is open in P if for every $p \in D$, for every $q \in P$, if $q \leq p$ then $q \in D$, i.e., D is downward closed. $A \subseteq P$ is an antichain if for every $p \in A$, for every $q \in A$, if $p \neq q$, then p and q are incompatible. An antichain is maximal if every $p \in P$ is compatible with some member of A, i.e., for every $p \in P$, there is $r \in A$ and there is $q \in P$ such that $q \leq r$ and $q \leq p$. $F \subseteq P$ is a filter if F is not empty, F is upward closed, i.e., if $p \in F$ and $q \in P$, and $p \leq q$, then $q \in F$, and for every $p \in F$, for every $q \in F$, there is $r \in F$ such that $r \leq p$ and $r \leq q$. Let N be a family of dense sets in P. A filter $G \subseteq P$ is generic over N if for every $D \in N$, $G \cap D \neq \emptyset$. A partially ordered set $(P, \leq)$ is (ω, ∞)-distributive if for every countable family F of dense open sets, the intersection of F is a dense subset of P. Sometimes, it is also called Baire, in particular, when referring to trees.

A tree T is a partially ordered set such that $\{t \in T \mid t <_T s\}$ is well-ordered under the tree ordering $<_T$ for all $s \in T$. For a tree T, for $t \in T$, the height of t, $ht(t)$, is the least ordinal α which is isomorphic to $\{s \in T \mid s <_T t\}$. The height of the tree T, $ht(T)$, is the least ordinal α such that for all $t \in T$, $ht(t) < \alpha$. For every $\alpha < ht(T)$, the αth level, denoted by T_α, is the set of all $t \in T$ such that $\alpha = ht(t)$. We use $T\restriction_\alpha$ to denote the set of all $t \in T$ such that $ht(t) < \alpha$. Notice that this is a subtree of T. A branch of T is a maximal linearly ordered subset of T; the length of a branch is its order type. A branch is cofinal if its length is the height of T. When we consider a tree as a forcing notion, we consider the reverse order of the tree ordering. Or sometimes we just simply consider the tree growing downward. This should not cause any confusion. For example, we say that a tree is a Baire tree if it is (ω, ∞)-distributive as a forcing notion. We will be mainly interested in trees of height ω_1 in this paper.

Trees have long been interesting objects of study in set theory. Given a tree, a very basic question is whether there is a cofinal branch. For trees of height ω_1, one of the first interesting examples is an Aronszajn tree (a tree of height ω_1, each level is countable, but there is no cofinal branch). Then come Souslin trees (trees of height ω_1, each level is countable and each maximal antichain is countable). Every Souslin tree is an Aronszajn tree. However, the existence of a Souslin tree is independent of the standard set theory, ZFC (see Jech [11] for references regarding these facts). Trees of height ω_1 are also divided into two classes: special trees and nonspecial trees. A special tree is a tree which is a union of countably many antichains. It is now known that under Martin's Axiom, every Aronszajn tree is special, while under the Proper Forcing Axiom, every tree of height ω_1 and cardinality $\aleph_1$ is special (see [2] and references there). Also it is known that in the constructible universe of Gödel, Souslin trees exist and there are Kurapa trees (trees of height ω_1 with each level countable, and there are $\aleph_2$ many cofinal branches). (See Jech [11] for references on these facts.)

Nonspecial trees have been extensively studied (see [17] and the references there). Notice that Baire trees are nonspecial trees. Todorcevic reformulated Rado's conjecture in terms of nonspecial trees in [16], namely, if a tree is nonspecial then it has a nonspecial subtree of size $\aleph_1$, and he derived many interesting consequences in [20]. Recently, we showed that Rado's conjecture also implies the nonstationary ideal on ω_1 is presaturated in [4].

All of these lead us to consider the following general question: Which trees of height ω_1 have cofinal branches?

We will introduce a class of trees, strongly Baire trees, in section 3. The strongly Baire property is a natural strengthening of the Baire property. Although there are Baire trees of height ω_1 without having cofinal branches, every strongly Baire tree may have a cofinal branch. In fact, we propose a cofinal branch principle which says that every strongly Baire tree of height ω_1 has a cofinal branch. It follows from the Semiproper Forcing Axiom. In section 4, we prove that this cofinal branch principle implies the Strong Reflection Principle holds, there is no Souslin tree, and $MA^+(\sigma\text{-closed})$ holds. In section 2, we present a connection between stationary sets and (ω, ∞)-distributivity.

2. Stationary sets, trees and (ω, ∞)-distributivity

Assume that κ is a regular uncountable cardinal and $\lambda \geq \kappa$ is a cardinal. Let A be a set of cardinality λ. We use $P_\kappa(A)$ to denote the set of all subsets of A of size $< \kappa$. A set $C \subseteq P_\kappa(A)$ is **closed** if for every $\subseteq$-increasing countable sequence

$\langle x_\alpha | \ \alpha < \theta \rangle$ from C of length $\theta < \kappa$ ($x_\alpha \subseteq x_\beta$ for all $\alpha < \beta < \theta$), the union of the sequence $\bigcup\{x_\alpha | \ \alpha < \theta\}$ is also in C. C is **unbounded** if for all $x \in P_\kappa(A)$ there exists some $y \in C$ such that $x \subseteq y$. C is a **club** if C is both closed and unbounded. A set $S \subseteq P_\kappa(A)$ is **stationary** if for every club C, the intersection $S \cap C$ is not empty.

THEOREM 2.1 (Jech [10]): (1) *All the clubs on $P_\kappa(A)$ generate a κ-complete normal filter. Namely, if $\langle C_a | \ a \in A \rangle$ is a sequence of sets closed and unbounded in $P_\kappa(A)$, then*

$$C = \{x \in P_\kappa(A) | \ \forall a \in x \ x \in C_a\}$$

is also closed and unbounded in $P_\kappa(A)$. C is called the diagonal intersection of the sequence.

(2) *If S is a stationary set in $P_\kappa(A)$, $f \colon S \to A$ satisfying that $f(x) \in x$ for all $x \in S$, then there exists a stationary $T \subseteq S$ such that f is constant on T.*

Another basic fact is that given two sets A and B of the same uncountable cardinality, then there is a natural correspondence between the closed and unbounded sets and stationary sets. Namely if $f \colon A \to B$ is a bijection, $C \subseteq P_\kappa(A)$ is a set closed and unbounded in $P_\kappa(A)$, letting $C^* = \{f''X | \ X \in C\}$, then C^* is a set closed and unbounded in $P_\kappa(B)$ (where $f''X$ denotes the set $\{f(a) | \ a \in X\}$). If S is a set stationary in $P_\kappa(A)$, letting $S^* = \{f''X | \ X \in S\}$, then S^* is a set stationary in $P_\kappa(B)$.

Assume that $B \subseteq A$. Let C be a set closed and unbounded in $P_\kappa(A)$. Then $\{X \cap B | \ X \in C\}$ is a set closed and unbounded in $P_\kappa(B)$. Conversely, if C is a set closed and unbounded in $P_\kappa(B)$, then $\{X \in P_\kappa(A) | \ X \cap B \in C\}$ is a set closed and unbounded in $P_\kappa(A)$. Hence, if S is a set stationary in $P_\kappa(A)$, then $\{X \cap B | \ X \in S\}$ is stationary in $P_\kappa(B)$. If S is stationary in $P_\kappa(B)$, then $\{X \in P_\kappa(A) | \ X \cap B \in S\}$ is stationary in $P_\kappa(A)$.

We will be interested mainly in the case that $\kappa = \omega_1$. For an uncountable set A, we use $[A]^\omega$ to denote the set of all infinite countable subsets of A. This is a set closed and unbounded in $P_{\omega_1}(A)$.

If $f \colon [A]^{<\omega} \to A$ and $X \subseteq A$, then we say that X is closed under f if X is nonempty and for all $e \in [X]^{<\omega}$, $f(e) \in X$. If $X \subseteq A$, the closure of X under f is the smallest $Y \supseteq X$ which is closed under f. Usually we denote the closure of X under f by $cl_f(X)$. Notice that given $f \colon [A]^{<\omega} \to A$, the family of all countable subsets of A which are closed under f is closed and unbounded in $[A]^\omega$. In particular, given an uncountable structure of countable language endowed with a well ordering, the set of all countable elementary submodels is closed and unbounded.

THEOREM 2.2 (Kueker [14]): *Assume that C is closed and unbounded in $[A]^\omega$. Then there exists an $f\colon [A]^{<\omega} \to A$ such that every countable subset X of A which is closed under f is in C. Hence $S \subseteq [A]^\omega$ is stationary if and only if for every $f\colon [A]^{<\omega} \to A$, there exists an $x \in S$ such that x is closed under f, i.e., $f(e) \in x$ for all $e \in [x]^{<\omega}$.*

Let κ be a regular cardinal $\geq \omega_2$. Let H_κ be the set of all sets hereditarily of size less than κ. We assume that H_κ is endowed with a well ordering. Notice that $[H_\kappa]^\omega \subseteq H_\kappa$. We are interested in such structures because $(H_\kappa, \in)$ is a model of ZFC minus the power set axiom (when κ is large, a sufficient initial segment of the power set axiom will hold). Essentially, when we choose large enough κ, $(H_\kappa, \in)$ will be a good model of set theory satisfying our needs. We are interested in the stationary sets on the spaces of countable sets, primarily because there is a closed and unbounded set of countable elementary submodels of $(H_\kappa, \in)$ and these countable models are rich enough to have many desired properties. For example, if N is a countable elementary submodel of H_κ, then $N \cap \omega_1$ is a countable ordinal. This follows from the fact that if $x \in N$ is countable then $x \subseteq N$.

We use $N \prec M$ to denote that N is an elementary submodel of M as usual.

Recall from [6] that a set $S \subseteq [H_\kappa]^\omega$ is projective stationary if for every stationary $A \subseteq \omega_1$, the set $\{N \in S|\ N \cap \omega_1 \in A\}$ is stationary. Projective stationary sets are natural objects which are between stationary sets and clubs. We will use projective stationary sets to define strongly Baire partially ordered sets. Before we do that, let us see some essential connections between stationary sets and the Baire property. This will motivate the concept of strongly Baire trees in the next section.

Let P be a partially ordered set. Let κ be a sufficiently large regular cardinal. $P \in N \prec H_\kappa$ is countable. A condition $p \in P$ is a **strong master condition for** N if for every dense $D \in N$, there exists a condition $q \in D \cap N$ such that $p \leq q$ (see [7]). Also, a sequence $\langle p_n|\ n < \omega\rangle$ is a generic sequence for N if for every dense $D \in N$, there is some $n < \omega$ such that $p_n \in D \cap N$ and $p_n \geq p_{n+1}$ for all $n < \omega$. Notice that for every countable model N, a generic sequence exists. Recall that a partially ordered set P is (ω, ∞)-distributive if the intersection of any countably many dense open sets is dense. We give below a characterization of (ω, ∞)-distributivity in term of stationary sets and strong master conditions.

THEOREM 2.3: *Let P be a partially ordered set. Then P is (ω, ∞)-distributive if and only if for every $p \in P$, for every sufficiently large regular cardinal κ, the set S_p defined by*

$N \in S_p$ if and only if $N \prec H_\kappa$ is countable, $p \in N$, and there exists some $q \in P$ such that $q \leq p$ and q is a strong master condition for N

is stationary in $[H_\kappa]^\omega$.

Consider the *cut-and-choose* game on P. Player I begins by selecting some $p \in P$ and a maximal antichain A_0 below p. Player II responds by choosing some $p_0 \in A_0$. At the nth move, I plays a maximal antichain A_n below p and II chooses $p_n \in A_n$. II wins the play if and only if there is a $q \in P$ such that $q \leq p_n$ for all $n < \omega$.

THEOREM 2.4 (Jech [12]): *P is (ω, ∞)-distributive if and only if I has no winning strategy.*

We apply this game and this theorem to prove the previous theorem.

Proof: $\Rightarrow$ Assume that P is (ω, ∞)-distributive. Let $p \in P$. Let λ be a sufficiently large regular cardinal. We show that S_p is stationary.

Let $f\colon [H_\lambda]^{<\omega} \to H_\lambda$ and C_f be the set of all elementary countable submodels which are closed under f. For each $N \in C_f$, we fix an enumeration $\langle A_n^N \mid n < \omega\rangle$ of all maximal antichains below p which are in N. Let $\pi\colon \omega \to \omega \times \omega$ be a paring function such that $(\pi(n))_0 \leq n$, where $\pi(n) = ((\pi(n))_0, (\pi(n))_1)$ for all $n < \omega$.

We define a strategy σ_f^p for I as follows:

Let $N_0 \in C_f$ be the closure of $\{p, P\}$ under f and the skolem functions. Let $A_0 = A_{(\pi(0))_1}^{N_{(\pi(0))_0}}$. At the 0th move, I plays p and A_0. For a given $p_0 \in A_0$, let $N_1 \in C_f$ be the closure of $N_0 \cup \{p_0\}$ under f and the skolem functions. Let $A_1 = A_{(\pi(1))_1}^{N_{(\pi(1))_0}}$. At the 1st move, I plays A_1. In general, let $p_n \in A_n$ be given. Let N_{n+1} be the closure of $N_n \cup \{p_n\}$ under f and the skolem functions and let $A_{n+1} = A_{(\pi(n+1))_1}^{N_{(\pi(n+1))_0}}$. At the $(n+1)$th move, I plays A_{n+1}.

This defines a strategy for I. At the end of each such play, when I follows this strategy, if N is the union of all N_n's, then $N \in C_f$ and $\langle A_n \mid n < \omega\rangle$ enumerates all maximal antichains below p which are in N. If II wins the play, then any witness will be a strong master condition for this model N. Since P is (ω, ∞)-distributive, by the quoted theorem above, I has no winning strategy. In particular, the strategy σ_f^t defined above cannot be a winning strategy. Therefore there is a play in which I follows the strategy but II wins the play.

This shows that S_p is indeed stationary.

$\Leftarrow$ Given $p \in P$ and a countable sequence $\langle D_n \mid n < \omega\rangle$ of dense open sets, let κ be a sufficiently large regular cardinal. Let S_p be the stationary sets of countable models with a strong master condition below p. Let $N \in S_p$ be such that the sequence of dense open sets is in N. Then any strong master condition for N below p witnesses that the intersection is not empty below p. Hence the intersection must be dense. ∎

A sequence $\langle N_\alpha |\ \alpha < \theta\rangle$ of countable submodels of H_κ is an **increasing continuous $\in$-chain** of length θ if for all $\alpha < \theta$ ($N_\alpha \prec H_\kappa$) and for all $\alpha < \beta < \theta$ ($N_\alpha \in N_\beta$), and if $\alpha < \theta$ is a limit ordinal, then $N_\alpha = \bigcup_{\beta<\alpha} N_\beta$.

An interesting fact about the stationary sets of countable models is that if $S \subseteq [H_\kappa]^\omega$ is stationary then S contains an increasing continuous $\in$-chain of length α for every countable ordinal α. We will need the following well-known lemma. For a proof, see [6] for example.

LEMMA 2.1: *If $S \subseteq [H_\kappa]^\omega$ is stationary, then for every countable ordinal α, there exists an increasing continuous $\in$-chain*

$$\langle N_\gamma |\ \gamma \leq \alpha\rangle$$

of length $\alpha + 1$ such that $N_\gamma \in S$ for all $\gamma \leq \alpha$.

There is a natural connection between stationary sets and Baire trees.

Let $\kappa \geq \omega_2$ be regular. Let $S \subseteq [H_\kappa]^\omega$. Associated with S is a tree U_S defined as follows: $t \in U_S$ if and only if there exists some countable ordinal α such that $t\colon \alpha+1 \to S$ and t is an increasing continuous $\in$-chain of length $\alpha+1$. Sometimes we refer to U_S as the **canonical tree of** S. Certainly U_S is a tree of height at most ω_1. In fact, if S is stationary, then U_S has height ω_1 as the above lemma indicates. Todorcevic in [20] showed that if S is stationary, then U_S is a Baire tree. It may be possible this is known to many other people. In that case, we just take [20] as one example of references. It is natural to ask if the converse would be true. Obviously there are some difficulties. Whether the tree U_S is Baire or not is a question relating only to the tree itself, while the question whether S is stationary is related to certain spaces. For instance, $S \subseteq [H_\kappa]^\omega$ may be stationary in $[H_\kappa]^\omega$ but certainly not stationary in $[H_{\kappa^+}]^\omega$. The corresponding tree U_S will be a Baire tree in either case. We need certain conditions to connect the tree U_S to the spaces under discussion, just as S does. This leads to a kind of density property essentially.

Let us call a tree T a normal tree if for every $t \in T$, for every $\alpha < ht(T)$, if $ht(t) < \alpha$, then there is an $s \in T$ such that $t \leq s$ and $ht(s) \geq \alpha$.

Notice that for $S \subseteq [H_\kappa]^\omega$, S is unbounded in $[H_\kappa]^\omega$ and U_S is a normal tree of height ω_1 if and only if for all $\alpha < \omega_1$, for all $x \in H_\kappa$, the following sets D_α and D_x are dense in U_S:

$$D_\alpha = \{t \in U_S |\ \alpha \leq ht(t)\},$$

$$D_x = \{t \in U_S |\ \exists \beta \le ht(t) x \in t(\beta)\}.$$

(We thank Professor T. Jech for suggesting that we use normal trees instead of the density property in formulating the following theorem.)

THEOREM 2.5: *Let $\kappa \ge \omega_2$ be a regular cardinal. Let $S \subseteq [H_\kappa]^\omega$ be unbounded in $[H_\kappa]^\omega$ and U_S be the canonical tree of S. Then the following are equivalent:*

(1) S is stationary in $[H_\kappa]^\omega$.

(2) U_S is a normal tree of height ω_1 and for every sufficiently large regular λ, the set S^* defined by

$N \in S^*$ if and only if $N \in [H_\lambda]^\omega$ and $S, U_S \in N$ and for all $\sigma \in N \cap U_S$ there exists a strong master condition $t \in U_S$ for N such that $t \le \sigma$

is stationary in $[H_\lambda]^\omega$.

(3) U_S is a normal tree of height ω_1 and U_S is a Baire tree.

(4) U_S is a normal tree of height ω_1 and for every $t \in U_S$, for every sufficiently large regular λ, the set S_t^* defined by

$N \in S_t^*$ if and only if $N \in [H_\lambda]^\omega$ and $S, U_S, t \in N$ and there exists a strong master condition $\sigma \in U_S$ for N such that $\sigma \le t$

is stationary in $[H_\lambda]^\omega$.

Proof: (1) $\Rightarrow$ (2). We assume that S is stationary in $[H_\kappa]^\omega$. First let us check that U_S has the density property with respect to H_κ. To see this, let $\alpha < \omega_1$ and $x \in H_\kappa$. Let $t \in U_s$.

Then the set $S_t = \{N \in S |\ \{t, x\} \subseteq N\}$ is stationary in $[H_\kappa]^\omega$. Let $\langle M_\gamma |\ \gamma \le \alpha\rangle$ be an increasing continuous ϵ-chain from S_t. We then define $\sigma(\beta) = t(\beta)$ for all $\beta \in dom(t)$ and $\sigma(\beta) = M_\beta$ for all β such that $dom(t) \le \beta \le \alpha$. Then we have that $\sigma \le t$ and $\sigma \in D_\alpha \cap D_x$.

Let λ be a sufficiently large regular cardinal. Let $f : [H_\lambda]^{<\omega} \to H_\lambda$. Let $N \prec H_\lambda$ be countable such that N is closed under f and $N \cap H_\kappa \in S$ and $\{H_\kappa, S, U_S\} \subseteq N$. Let $\delta = N \cap \omega_1$ and $t \in N \cap U_S$. Let $\langle D_n |\ n < \omega\rangle$ be a list of all dense subsets of U_S which are in N. Starting from $t_0 = t$, inductively pick $t_{n+1} \in D_n \cap N$ so that $t_{n+1} \le t_n$. Let θ_n be the largest countable ordinal in $dom(t_n)$. By elementarity and a density argument, we have $\delta = \bigcup_{n<\omega} \theta_n$ and $N \cap H_\kappa = \bigcup_{n<\omega} t_n(\theta_n)$. Therefore,

$$\sigma = \bigcup_{n<\omega} t_n \cup \{(\delta, N \cap H_\kappa)\}$$

is in U_S, a condition stronger than all t_n. Hence σ is a strong master condition for N which is closed under f. Therefore, S^* is stationary in $[H_\lambda]^\omega$.

(2) $\Rightarrow$ (3). This follows from the direction $\Leftarrow$ of the previous theorem.

(3) $\Rightarrow$ (4). This follows from the direction $\Rightarrow$ of the previous theorem.

(4) $\Rightarrow$ (1). Let $f\colon [H_\kappa]^{<\omega} \to H_\kappa$. Let λ be a sufficiently large regular cardinal. Let $N \prec H_\lambda$ be countable such that f, S, U_S, H_κ are all in N and N has a strong master condition $\sigma \in U_S$. Let $\alpha = N \cap \omega_1$. Since U_S has the density property with respect to H_κ, by elementarity and a density argument, being a strong master condition for N, we must have $\sigma(\alpha) = N \cap H_\kappa$. Hence, $\sigma(\alpha)$ must be closed under f and $\sigma(\alpha) \in S$.

This shows that S is stationary in H_κ. ∎

3. Strongly Baire trees and a cofinal branch principle

In this section, we first give a natural strengthening of the (ω, ∞)-distributivity in the direction of Theorem 2.3 of the previous section. Namely, we will define the strongly Baire property, and then give some characterizations, including a game theoretic one.

Definition 3.1: Let P be a partially ordered set. We say that P is **strongly Baire** if for every sufficiently large regular cardinal κ, for every $p \in P$, the set S_P^p defined by

$N \in S_P^p$ if and only if $N \prec H_\kappa$ is countable and $p \in N$ and there exists some $q \in P$ such that $q \leq p$ and q is a strong master condition for N

is projective stationary in $[H_\kappa]^\omega$.

Certainly, every strongly Baire partially ordered set is (ω, ∞)-distributive. However, the converse is not true, as the following example indicates.

Example 3.1: Let A, B be two disjoint stationary subsets of ω_1. Let T be the tree of all countable closed subsets of A, ordered by end extension. This tree is a Baire tree of height ω_1 (see [13], for example) but not strongly Baire. And this tree has no cofinal branch.

It turns out these are the only counterexamples.

THEOREM 3.1: *P is strongly Baire if and only if P is (ω, ∞)-distributive and forcing with P preserves stationary sets of ω_1.*

Proof: $\Rightarrow$ P certainly is (ω, ∞)-distributive by Theorem 2.3.

Let $S \subseteq \omega_1$ be stationary. Let $\dot{C}$ be a name for a club in ω_1. Let $t \in T$ be a condition. Let N be a countable submodel containing $t, S, \dot{C}$ and $N \cap \omega_1 \in S$ and N has a strong master condition $\sigma \leq t$. Then $\sigma \Vdash N \cap \omega_1 \in \dot{C}$.

$\Leftarrow$ Assume that P is (ω,∞)-distributive and forcing with P preserving stationary subsets of ω_1. We shall prove that P is strongly Baire.

First, let us prove the following lemma which will also be used later.

LEMMA 3.1: *Assume that P is (ω,∞)-distributive. Let κ be a sufficiently large regular cardinal. Let $A = H_\kappa$ and $B = [H_\kappa]^\omega$. Then there is a name $\dot{f}$ for a function from $[A]^{<\omega}$ to A such that*

$$\Vdash \forall N \in [A]^\omega = B,\ N \prec A,\ N \text{ is closed under } \dot{f} \Rightarrow \dot{G} \text{ is generic over } N.$$

Proof: Let G be a P-generic filter over V. In $V[G]$, $[A]^\omega = B$, as the forcing is (ω,∞)-distributive.

CLAIM: *$\{N \in [A]^\omega \mid N \prec A, G$ is generic over $N\}$ contains a closed and unbounded subset of $[A]^\omega$.*

Assume otherwise. Then

$$S = \{N \in [A]^\omega \mid N \prec A,\ \exists \text{ a dense set } D \in N (G \cap D \cap N = \emptyset)\}$$

is stationary in $[A]^\omega$. So there is some dense $D \in A$ such that

$$S^* = \{N \in [A]^\omega \mid N \prec A,\ D \in N (G \cap D \cap N = \emptyset)\}$$

is stationary in $[A]^\omega$.

Let $\dot{S}$, $\dot{D}$ be names for these objects and let $q \in G$ be a condition that forces the above property of the objects.

Back to the ground model V. Let $p \leq q$ be a condition such that for some dense set D, p forces that $\dot{D} = D$. Let $p_0 \in D$ be such that $p_0 \leq p$. Then $p_0 \Vdash p_0 \in \dot{D} \cap \dot{G}$.

Let C be a closed and unbounded set of models N such that every member of C contains p_0, D, and $\dot{D}$.

Let G be a generic filter over V such that $p_0 \in G$. In $V[G]$, C is still closed and unbounded. Let $N \in C \cap \dot{S}/G$. Let $\dot{N}$ be a name for it and let $p_1 \in G$ be such that $p_1 \leq p_0$ and p_1 forces $\dot{N} \in C \cap \dot{S}$.

Back to V. Get some $N \in C$ and some $p_2 \leq p_1$ so that $p_2 \Vdash \dot{N} = N$. Then $p_2 \Vdash p_0 \in N \cap \dot{D} \cap \dot{G}$ and $p_2 \Vdash \dot{D} \in \dot{N} \in \dot{S}$ and $p_2 \Vdash \dot{G} \cap \dot{N} \cap \dot{D} = \emptyset$. This is a contradiction.

This finishes the proof of the lemma. ∎

We proceed to prove the theorem.

Let $T \subseteq \omega_1$ be stationary. Let g: $[H_\kappa]^{<\omega} \to H_\kappa$ be a function. Let $p \in P$.

Let $p \in G$ be a generic filter over V. In $V[G]$, T remains stationary. So there is some countable $N \prec A$ such that N is closed under g, $N \cap \omega_1 \in T$, and there is a sequence $\langle p_n |\ n < \omega\rangle$ from G such that $\langle p_n |\ n < \omega\rangle$ is a P-generic sequence for N, by the previous lemma.

Let $\dot{N}, \dot{\alpha}, \dot{s}$ be names and let $q \in G$ be such that $q \leq p$ and q forces that $\dot{N}$ is a countable elementary submodel of A, $\dot{N}$ is closed under the function g, $\dot{\alpha} = \dot{N} \cap \omega_1$, $\dot{\alpha} \in T$, $\dot{s}\colon \omega \to P \cap \dot{N}$ is a generic sequence for $\dot{N}$ and for all $n < \omega$, $\dot{s}(n) \in \dot{G}$.

Back to V. By (ω, ∞)-distributivity, let $r \leq q$ be a condition, and let $N \in B$, $\alpha \in \omega_1$, $t\colon \omega \to N \cap P$ be such that

$$r \Vdash \dot{N} = N,\ \dot{\alpha} = \alpha,\ \dot{s} = t.$$

So N is closed under g, $\alpha = N \cap \omega_1$, $\alpha \in T$ and t is a generic sequence over N, and $r \Vdash \forall n < \omega t(n) \in \dot{G}$. Therefore, r is a strong master condition for N.

This finishes the proof. ∎

We now give a game theoretic characterization of a strongly Baire partially ordered set.

Given a partially ordered set P, consider the following game $\mathcal{G}(P)$:

Player I starts the game by selecting a condition $p \in P$, and a stationary subset $A \subseteq \omega_1$.

At the nth move, Player I plays a name for an ordinal, $\dot{\alpha}_n$; Player II responds by choosing an ordinal γ_n and a countable ordinal β_n.

At the end of ω many steps, II wins the game if and only if there exists a condition $q \leq p$ such that $\forall n < \omega q \Vdash \dot{\alpha}_n = \gamma_n$ and, letting

$$\delta = \bigcup\{\gamma_n |\ n < \omega \& \gamma_n < \omega_1\},$$

if $\delta \geq \bigcup_{n<\omega} \beta_n$, then $\delta \in A$.

THEOREM 3.2: *P is strongly Baire if and only if I does not have a winning strategy in the game $\mathcal{G}(P)$.*

Proof: $\Rightarrow$ Assume that P is strongly Baire. We show that I has no winning strategy.

Given a name for an ordinal, there corresponds a maximal antichain. Let us fix such a correspondence f. Let κ be a sufficiently large regular cardinal such that every relevant parameter occurs in H_κ. Assume that I has a winning strategy σ. We are going to derive a contradiction. Let p and A be the condition and the stationary subset initiated by σ. Let $N \prec H_\kappa$ be a countable model such that $\{\sigma, P, f\} \subseteq N$, and $N \cap \omega_1 \in A$, and N has a strong master condition $q \leq p$. Being strongly Baire, we have such N. Let $\langle W_n | \ n < \omega \rangle$ be a list of all maximal antichains in N. Let $\langle \beta_n | \ n < \omega \rangle$ be an enumeration of $N \cap \omega_1$. At the ith move, II responds as follows: let $W_j = f(\dot{\alpha}_i)$ be the corresponding partition associated to the ordinal name $\dot{\alpha}_i$, played by I according to σ, let $p_j \in W_j \cap N$ be such that $p_j \geq q$, and let γ_i be the ordinal γ such that $p_j \Vdash \dot{\alpha}_i = \gamma$. II then plays γ_i, β_i. Certainly this game can continue ω many steps. At the end of the play, $q \Vdash \dot{\alpha}_n = \gamma_n$ for all $n < \omega$ and $N \cap \omega_1 = \bigcup_{n<\omega} \beta_n$, and $\delta = \bigcup \{\gamma_n | \ \gamma_n < \omega_1\} \leq N \cap \omega_1$, if $\delta = N \cap \omega_1$, then $\delta \in A$. Hence II wins the game. But I follows the strategy σ which is supposed to be a winning strategy for I. This is a contradiction.

$\Leftarrow$ Assume that I does not have a winning strategy. We want to show that P is strongly Baire.

For each countable ordinal $\alpha < \omega_1$, let us fix a maximal antichain B_α to represent the canonical name for α. And given a maximal antichain, there corresponds a name for an ordinal. Let us fix such a mapping g such that $g(B_\alpha)$ is the canonical name for α.

Let $p \in P$. Let $T \subseteq \omega_1$ be stationary.

Let λ be a sufficiently large regular cardinal. Let f: $[H_\lambda]^{<\omega} \to H_\lambda$ and C_f be the set of all elementary countable submodels which are closed under f and p, g, T are members. For each $N \in C_f$, we fix an enumeration $\langle A_n^N | \ n < \omega \rangle$ of all maximal antichains below p which are in N. Let π: $\omega \to \omega \times \omega$ be a paring function such that $(\pi(n))_0 \leq n$.

We define a strategy $\sigma_f^{(p,T)}$ for I as follows:

Let $N_0 \in C_f$ be the closure of $\{p, T, g, P, \langle B_\alpha | \ \alpha < \omega_1 \rangle\}$ under f and the skolem functions. I starts the game by playing (p, T). Let $A_0 = A_{(\pi(0))_1}^{N_{(\pi(0))_0}}$. Let $\dot{\alpha}_0 = g(A_0)$. At the 0th move, I plays $\dot{\alpha}_0$. For a given ordinal γ_0 and a countable ordinal β_0, let $N_1 \in C_f$ be the closure of $N_0 \cup \{\gamma_0, \beta_0\}$ under f and the skolem functions. Let $A_1 = A_{(\pi(1))_1}^{N_{(\pi(1))_0}}$ and let $\dot{\alpha}_1 = g(A_1)$. At the 1st move, I plays $\dot{\alpha}_1$. In general, let γ_n, β_n be given. Let N_{n+1} be the closure of $N_n \cup \{\gamma_n, \beta_n\}$ under f and the skolem functions and let $A_{n+1} = A_{(\pi(n+1))_1}^{N_{(\pi(n+1))_0}}$ and let $\dot{\alpha}_{n+1} = g(A_{n+1})$. At the $(n+1)$th move, I plays $\dot{\alpha}_{n+1}$.

This defines a strategy for I. At the end of each such play, when I follows this strategy, if N is the union of all N_n's, then $N \in C_f$ and $\langle A_n | \ n < \omega \rangle$ enumerates all maximal antichains below p which are in N. If II wins the play, then any witness will be a strong master condition for this model N. Also, because the canonical names for countable ordinals $< N \cap \omega_1$ are in N and $B_\alpha \in N$ for all $\alpha < N \cap \omega_1$, all the ordinals smaller than $N \cap \omega_1$ must be played as some γ_n by II at some stage n. Therefore

$$\delta = \bigcup \{\gamma_n | \gamma_n < \omega_1\} = N \cap \omega_1.$$

So $\delta \geq \bigcup_{n<\omega} \beta_n$. That II wins the play implies that $N \cap \omega_1 = \delta \in T$. Since I has no winning strategy, the strategy $\sigma_f^{(p,T)}$ defined above cannot be a winning strategy. Therefore there is a play in which I follows the strategy but II wins the play.

This shows that P is indeed strongly Baire. ∎

Remark: Let us notice that for every partially ordered set P, II has no winning strategy for the game. Let σ be a strategy for II. Let N be a countable elementary model of some H_κ for some sufficiently large regular cardinal κ such that P, σ are all in N. Let T_1 and T_2 be two disjoint stationary subsets of ω_1 which are in N. Consider two plays in which I enumerates all names for ordinals in N but initiates with T_1 and T_2 respectively. II plays according to the strategy σ. As N is closed under σ, and all names for ordinals smaller than $N \cap \omega_1$ are enumerated, II can win at most one of the plays, not both of the plays, following σ. Hence σ is not a winning strategy for II.

We are mainly interested in strongly Baire trees. It seems that being strongly Baire is a reasonable property to imply the existence of a cofinal branch for trees of height ω_1. However, being Baire is too weak, as we have seen from the previous example. Keeping this is in mind, it seems natural for us to consider the following cofinal branch principle.

Cofinal Branch Principle (CBP):

Every strongly Baire tree of height ω_1 has a cofinal branch.

Notice that the Cofinal Branch Principle can be restated as follows:

If T is a tree of height ω_1 and forcing with T preserves stationary sets of ω_1, then T has a cofinal branch.

This is based on the fact that if forcing with a tree T of height ω_1 preserves stationary sets of ω_1, then T is (ω, ∞)-distributive.

This cofinal branch principle follows from the Semiproper Forcing Axiom. Recall that a partially ordered set $(P, \leq)$ is semiproper if for sufficiently large regular cardinal κ, there is a function $f : [H_\kappa]^{<\omega} \to H_\kappa$ such that if $N \prec H_\kappa$ is closed under f, then for every $p \in P \cap N$, there is a condition $q \leq p$ such that if $\dot{\alpha} \in N$ is a name for a countable ordinal, then $q \Vdash \dot{\alpha} \in N$. The Semiproper Forcing Axiom is the following statement:

> If P is a semiproper partially ordered set, $\langle D_\alpha | \alpha < \omega_1 \rangle$ is a sequence of dense sets in P, then there is a filter $G \subseteq P$ such that $G \cap D_\alpha$ is not empty for every $\alpha < \omega_1$.

THEOREM 3.3: SPFA *implies* CBP.

Proof: Let T be a strongly Baire tree. By the previous theorem, forcing with T preserves stationary sets of ω_1. By Shelah's theorem, T is semiproper under the Semiproper Forcing Axiom. Applying SPFA, we conclude that T has a cofinal branch. ∎

4. Some consequences of CBP

In this section, we derive some consequences of the Cofinal Branch Principle.

First let us recall from [6] that the Projective Stationary Reflection Principle states for regular $\kappa \geq \omega_2$, that if $S \subseteq [H_\kappa]^\omega$ is projective stationary, then there is an increasing continuous sequence $\langle N_\alpha | \alpha < \omega \rangle$ of countable models from S.

It is equivalent to Todorcevic's Strong Reflection Principle. Hence it has many interesting consequences (see [6, 3] for more on this).

THEOREM 4.1: CBP *implies the Projective Stationary Reflection principle.*

Proof: Let $\kappa \geq \omega_2$ be a regular cardinal. Let $S \subseteq [H_\kappa]^\omega$ be a projective stationary set. Let T_S be the canonical tree associated with S.

CLAIM: *T_S is strongly Baire.*

Given this claim, by CBP, T_S has a cofinal branch. Any cofinal branch gives a continuous increasing $\in$-chain from S of length ω_1.

We now proceed to prove the claim.

Let λ be a sufficiently large regular cardinal. Let f: $[H_\lambda]^{<\omega} \to H_\lambda$ and $A \subseteq \omega_1$ be stationary. Since S is projective stationary, we can find a countable $N \prec H_\lambda$ such that N is closed under f, $N \cap \omega_1 \in A$, and $N \cap H_\kappa \in S$ and N contains all the parameters needed. Starting with any $t \in T_S \cap N$, inductively construct a countable sequence $\langle t_n | n < \omega \rangle$ so that $t_{n+1} \leq t_n$ and $t_{n+1} \in N \cap D_n$, where D_n is the nth dense open subset of T_S in N. Let $\delta = N \cap \omega_1$. Define

$$\sigma = \bigcup_{n<\omega} t_n \cup \{(\delta, N \cap H_\kappa)\}.$$

Then σ is a desired strong master condition for N.

This shows that T_S is a strongly Baire tree. ∎

Remark: A version of Theorem 2.5 in this context is actually true. Namely, for unbounded $S \subseteq [H_\kappa]^\omega$, let T_S be the canonical tree associated with S. Then S is projective stationary if and only if T_S is a normal tree of height ω_1 and T_S is strongly Baire.

COROLLARY 4.1: *Assume the Cofinal Branch Principle. Then the nonstationary ideal on ω_1 is saturated, $2^{\aleph_0} = \aleph_2$, and the Singular Cardinal Hypothesis holds.*

Proof: By Todorcevic [19], the Strong Reflection Principle, therefore CBP, implies that $\kappa^{\aleph_1} = \kappa$ for all regular $\kappa > \aleph_1$. By Silver's theorem this gives the Singular Cardinal Hypothesis as well as $2^{\aleph_0} \leq \aleph_2$. By a result of Gregory [9], CBP implies that $2^{\aleph_0} = 2^{\aleph_1}$. (I take this opportunity to thank the referee for pointing this out to me.) Also see Velickovic [21], Woodin [22] as well as Feng–Jech [6]. ∎

THEOREM 4.2: *Every Souslin tree is strongly Baire.*

Proof: Let T be a Souslin tree. Let κ be a sufficiently large regular cardinal. Let $N \prec H_\kappa$ be a countable elementary submodel such that $T \in N$. Let $\alpha = N \cap \omega_1$. Then every member of the maximal antichain T_α is a strong master condition for N. This is because if $t \in T_\alpha$, $A \in N$ is a maximal antichain, then $A \subseteq N$ and any condition in A which is compatible with t must be an initial segment of t. ∎

THEOREM 4.3: CBP *implies there is no Souslin tree.*

Proof: This follows from the previous theorem and CBP. If there were a Souslin tree T, then T would have a cofinal branch by CBP. ∎

Recall that MA^+(σ-closed) is the statement that if P is a σ-closed forcing (every countable decreasing sequence has a lower bound), $\dot{S}$ is a name for a stationary subset of ω_1, $\langle D_\alpha |\ \alpha < \omega_1\rangle$ is a sequence of dense sets in P, then there is a filter $G \subseteq P$ such that $G \cap D_\alpha$ is not empty for all $\alpha < \omega_1$, and

$$\{\alpha < \omega_1 |\ \exists p \in G p \Vdash \alpha \in \dot{S}\}$$

is stationary in ω_1. Shelah in [15] proved that the Semiproper Forcing Axiom implies MA^+(σ-closed).

We now show that the cofinal branch principle implies $MA^+(\sigma\text{-closed})$. The basic part of the proof is due to Shelah. We use trees. The main idea is to code needed generic objects by branches of a tree.

THEOREM 4.4: CBP *implies* $MA^+(\sigma\text{-closed})$.

Proof: Let P be a σ-closed forcing notion. Let $\dot{S}$ be a name for a stationary subset of ω_1. Let $\langle D_\alpha | \alpha < \omega_1\rangle$ be a sequence of dense subsets of P. For each $\alpha < \omega_1$, let P_α, $\dot{S}_\alpha$ and D^α_β for $\beta < \omega_1$ be an isomorphic copy. Let P^* be the product of the P_α's with countable support. Let $\dot{S}^*$ be a P^* name for the diagonal union of the $\dot{S}_\alpha$'s. For each $p \in P$, we use $spt(p)$ to denote the support of p. Namely,

$$spt(p) = \{\alpha < \omega_1 | \, p(\alpha) \neq \mathbf{1}\}.$$

LEMMA 4.1 (Shelah [15]): *Assume that the nonstationary ideal on ω_1 is saturated. Then there is a stationary subset $A \subseteq \omega_1$ such that $\omega_1 - A$ is stationary and for every stationary $B \subseteq \omega_1 - A$, $P^* \Vdash B \cap \dot{S}^*$ is stationary.*

By Corollary 4.1, the nonstationary ideal on ω_1 is saturated under CBP. Let us fix a stationary set $A \subseteq \omega$ as given by Shelah's lemma.

We define a tree T as follows:

$(f, g) \in T$ if and only if (1) for some countable ordinal α, $f\colon \alpha + 1 \to P^*$ and $g\colon \alpha + 1 \to \omega_1$, (2) g is strictly increasing and continuous, and (3) for $\beta < \beta' \leq \alpha$, $f(\beta') \leq f(\beta)$, and (4) for $\beta \leq \alpha$, $g(\beta) < \sup(spt(f(\beta)))$ and $f(\beta) \Vdash g''(\beta + 1) \subseteq A \cup \dot{S}^*$, and for all $\gamma \leq g(\beta)$, for all $\delta \in spt(f(\beta))$ either $f(\beta)(\delta) \Vdash \gamma \in \dot{S}_\delta$ or $f(\beta)(\delta) \Vdash \gamma \notin \dot{S}_\delta$, and there is some $p \in D^\delta_\gamma$ such that $f(\beta)(\delta) \leq p$.

The order of T (growing downward) is by extension.

LEMMA 4.2: *T has height ω_1.*

LEMMA 4.3: *T is strongly Baire.*

We shall prove these two lemmas by proving several lemmas in the following. Given these two lemmas, we can finish the proof of the theorem as follows.

Let $\langle (f_\alpha, g_\alpha) | \, \alpha < \omega_1\rangle$ be a cofinal branch of the tree. Let $g = \bigcup_{\alpha<\omega_1} g_\alpha$. Let $f = \bigcup_{\alpha<\omega_1} f_\alpha$. Then $g\colon \omega_1 \to \omega_1$ is a strictly increasing continuous function. Let C be the range of g. Then C is a closed and unbounded subset of ω_1. $f\colon \omega_1 \to P^*$ is a sequence of comparable conditions. $(\alpha < \beta < \omega_1 \Rightarrow f(\alpha) \geq f(\beta).)$

Let $G = \{p \in P^* | \, f(\alpha) \leq p \text{ for some } \alpha < \omega_1\}$ and, for each $\alpha < \omega_1$, let

$$G_\alpha = \{p \in P_\alpha | \, f(\beta)(\alpha) \leq_\alpha p \text{ for some } \beta < \omega_1\}.$$

Then $C \subseteq A \cup \dot{S}^*[G]$. Therefore, there is some $\alpha < \omega_1$ such that $\dot{S}_\alpha[G_\alpha]$ is stationary and G_α meets all the dense sets D^α_β for $\beta < \omega_1$. ∎

We now proceed to prove the two lemmas.

LEMMA 4.4: *For $p \in P^*$, and $\rho < \sup(spt(p))$, the set $D_{\rho,p}$ defined by*

$r \in D_{\rho,p}$ if and only if $r \in P^*$, $r \leq p$, and for all $\beta < \sup(spt(p))$, for all $\gamma \leq \rho$, there is some $s \in D^\beta_\gamma$ such that $s \geq r(\beta)$, and either $r(\beta) \Vdash \gamma \in \dot{S}_\beta$ or $r(\beta) \Vdash \gamma \notin \dot{S}_\beta$

is dense below p.

Proof: It follows from the σ-closeness.

LEMMA 4.5: *Assume that $p \in P^*$ and x is a closed bounded subset of ω_1. Assume that $p \Vdash x \subseteq A \cup \dot{S}^*$ and that $\max(x) < \sup(spt(p))$. Then there is some $q \in P^*$ such that $q \leq p$, $\sup(spt(q)) > \sup(spt(p))$ and for all $\beta < \sup(spt(q))$, for all $\gamma \leq \max(x)$, either $q(\beta) \Vdash \gamma \in \dot{S}_\beta$ or $q(\beta) \Vdash \gamma \notin \dot{S}_\beta$ and there is some $s \in D^\beta_\gamma$ such that $s \geq q(\beta)$.*

(We will call such q a **decisive condition** with respect to x.)

Proof: Let $p \in P^*$. Let x be a closed bounded subset of ω_1. Let $\delta = \max(x)$. Let $p_0 \in D_{\delta,p}$. Inductively, let $p_{n+1} \in D_{\delta,p_n}$ be such that $\sup(spt(p_{n+1})) > \sup(spt(p_n))$. Let $y = \bigcup_{n<\omega} spt(p_n)$. Then pick a condition $q \in P^*$ so that $y = spt(q)$ and for all $\beta \in spt(q)$, $q(\beta) = \bigwedge_{n<\omega} p_n(\beta)$. ∎

LEMMA 4.6: *For all $\alpha < \omega_1$, $E_\alpha = \{(f, x) \in T \mid \alpha < dom(f) = dom(x)\}$ is dense in T.*

Proof: Let $\alpha < \omega_1$ and $(f, x) \in T$. Let δ be such that $dom(f) = dom(x) = \delta + 1$. Let $f^* = f(\delta)$.

Let G be a P^* generic over V such that $f^* \in G$. In $V[G]$, $A \cup \dot{S}^*/G$ is stationary in ω_1. Let $b \subseteq A \cup \dot{S}^*/G$ be a closed bounded subset such that $g(\delta) < \min(b)$ and $o.t.(b) > \alpha$. Let $\dot{b}$ a name for such a b and let $p \in G$ be such that $p \leq f^*$ and

$$p \Vdash g''(\delta+1) \cup \dot{b} \subseteq A \cup \dot{S}^*, \& g(\delta) < \min(\dot{b}), \& \alpha < o.t.(\dot{b}),$$

and

$$p \Vdash \dot{b} \text{ is closed and bounded.}$$

Back to the ground model V. Let b be a closed bounded subset of ω_1 and let $p_0 \leq p$ be such that $p_0 \Vdash b = \dot{b}$ and $\max(b) < \sup(spt(p_0))$. Then let $q \leq p_0$ be a decisive condition with respect to $g''(\delta + 1) \cup b$ given by the previous lemma.

Let $\eta+1 = o.t.(b)$. Define $g_1\colon \delta+1+\eta+1 \to \omega_1$ by $g_1\restriction_{\delta+1} = g$ and $g_1\colon [\delta+1, \eta] \to b$ is the canonical order isomorphism. Then define $f_1\colon \delta + 1 + \eta + 1 \to P^*$ by $f_1\restriction_{\delta+1} = f$ and for all $\beta \in [\delta, \eta]$, $f_1(\beta) = q$. Then $(f_1, g_1) \in T$ and $(f_1, g_1) \in E_\alpha$.

This shows that E_α is dense in T. ∎

It follows that the tree T has height ω_1.

Proof of Lemma 4.3: Let $(f, x) \in T$ be a condition. Let κ be a sufficiently large regular cardinal. Let $B \subseteq \omega_1$ be stationary and let $h\colon [H_\kappa]^{<\omega} \to H_\kappa$. We need to find a countable elementary model $N \prec H_\kappa$ such that $(f, x) \in N$, and N is closed under h, $N \cap \omega_1 \in B$ and N has a strong master condition $(g, y) \leq (f, x)$.

CASE 1: $B \subseteq A$.

Let $N \prec H_\kappa$ be a countable model such that $N \cap \omega_1 \in B$, N is closed under h, and $(f, x) \in N$. Let $\langle D_n | \, n < \omega\rangle$ be an enumeration of all dense subsets of T which are in N. Inductively pick $(f_{n+1}, x_{n+1}) \in D_n \cap N$ such that $(f_{n+1}, x_{n+1}) \leq (f_n, x_n)$ with $f_0 = f$ and $x_0 = x$. Let $\delta = N \cap \omega_1$ and $y = \bigcup_{n<\omega} x_n \cup \{(\delta, \delta)\}$. Let $r = \bigcup_{n<\omega} f_n$. Then $r\colon \delta \to P^*$ is such that $r(\beta) \leq r(\beta')$ if $\beta' < \beta < \delta$. Let $r^* \leq \bigwedge_{\beta<\delta} r(\beta)$ be such that $\delta + 1 \in spt(r^*)$. Then $r^* \Vdash y''(\delta+1) \subseteq A \cup \dot{S}^*$ since $\delta \in B \subseteq A$.

Let $q \in P^*$ be a decisive condition with respect to the range of y such that $q \leq r^*$ given by the decisive condition lemma. Then we define $g\colon \delta + 1 \to P^*$ by $g\restriction_\delta = r$ and $g(\delta) = q$. Then $(g, y) \in T$ and $(g, y) \leq (f, x)$ is a strong master condition for N.

CASE 2: $B \subseteq \omega_1 - A$.

In this case, $P^* \Vdash B \cap \dot{S}^*$ is stationary.

LEMMA 4.7: *Let $p \in P^*$. Then the set U_p defined by*

$N \in U_p$ if and only if $N \prec H_\kappa$ is countable, $p \in N$, and there is a strong master condition $q \in P^*$ for N such that $q \leq p$ and $q \Vdash N \cap \omega_1 \in B \cap \dot{S}^*$

is stationary in $[H_\kappa]^\omega$.

First notice that for a countable model N, if p is a strong master condition for N, and if p does not force $N \cap \omega_1 \notin B \cap \dot{S}^*$, then there is a stronger q which forces that $N \cap \omega_1 \in B \cap \dot{S}^*$ and q is a strong master condition for N.

Proof: Assume the lemma does not hold. Then for some condition $p \in P^*$ the set X_p of all countable elementary submodels N such that $p \in N$, and every strong master condition $q \leq p$ for N forces $N \cap \omega_1 \notin B \cap \dot{S}^*$ must contain a closed and unbounded set in $[H_\kappa]^\omega$. Let $h\colon [H_\kappa^{<\omega} \to H_\kappa$ be such that $C_h \subseteq X_p$. Let $W = H_\kappa$ and $U = [H_\kappa]^\omega$.

Let $p \in G$ be a generic filter over V. In $V[G]$, there is a countable model $N \in C_h$ such that $N \prec W$, $p \in N$, $N \cap \omega_1 \in B \cap \dot{S}^*/G$, and G is P^* generic over N by Lemma 3.1. Let $\dot{N}$, $\dot{t}$, $\dot{\alpha}$ be names for the objects ($\dot{t}\colon \omega \to \dot{N} \cap P^*$, $\dot{\alpha} = \dot{N} \cap \omega_1$) and let $r \in G$ be a condition to force the facts on these objects.

Back to the ground model V. Let $q \in P^*$ be stronger than r such that for some $N \in C_h$, and some sequence $t\colon \omega \to N \cap P^*$, and some countable ordinal α, we have

$$q \Vdash \dot{N} = N,\ \dot{\alpha} = \alpha = N \cap \omega_1, \forall n < \omega t(n) = \dot{t}(n).$$

Then $q \leq p$ is a strong master condition for $N \in C_h$ and $q \Vdash N \cap \omega_1 \in B \cap \dot{S}^*$. This is a contradiction.

This finishes the proof of the lemma. ∎

For each $(f,x) \in T$, let f^* be the last condition of f, namely, if $\delta + 1 = dom(f) = dom(x)$, then $f^* = f(\delta)$.

Let $(f,x) \in T$ and $h\colon [H_\kappa]^{<\omega} \to H_\kappa$. Let $N \prec H_\kappa$ be countable such that N is closed under h, $(f,x) \in N$ and N has a strong master condition $p \leq f^*$ that forces $N \cap \omega_1 \in B \cap \dot{S}^*$.

Let $\langle D_n |\ n < \omega\rangle$ be an enumeration of all dense open sets of T which are in N.

Let $C_0 = \{(g,y) \in D_0 |\ (g,y) \leq (f,x)\}$.

Let $C_0^* = \{g^* |\ \exists y(g,y) \in C_0\}$. Then $C_0^* \in N$ is dense below f^*. Let $g^* \in C_0^* \cap N$ be such that $g^* \geq p$. Then take some g_1, y_1 so that $(g_1, y_1) \in C_0 \cap N$ and $g_1^* \geq p$.

Inductively, define

$$C_n = \{(g,y) \in D_n |\ (g,y) \leq (g_n, y_n)\}$$

and

$$C_n^* = \{g^* |\ \exists y(g,y) \in C_n\}.$$

Then $C_n \in N$ and $C_n^* \in N$ is dense below g_n^*. Let $g^* \in C_n^* \cap N$ be such that $g^* \geq p$. Then take some g_{n+1}, y_{n+1} so that $(g_{n+1}, y_{n+1}) \in C_n \cap N$ and $g_{n+1}^* \geq p$.

Let $r = \bigcup_{n<\omega} g_n$ and let $y = \bigcup_{n<\omega} y_n \cup \{(N \cap \omega_1, N \cap \omega_1)\}$.

Then $p \leq \bigwedge_{\beta < N \cap \omega_1} r(\beta)$ and $p \Vdash y \subseteq A \cup \dot{S}^*$.

Let $q \leq p$ be a decisive condition with respect to the range of y. Define g by $g\restriction_{N \cap \omega_1} = r$ and $g(N \cap \omega_1) = q$. Then $(g,y) \in T$ and $(g,y) \leq (f,x)$, and (g,y) is a strong master condition for N. As $g^* \Vdash N \cap \omega_1 \in B \cap \dot{S}^*$, $N \cap \omega_1 \in B$.

This finishes the proof. ∎

5. Some remarks

In an unpublished work, Woodin showed that one could force Todorcevic's Strong Reflection Principle from a supercompact cardinal by revised countable support while preseving Souslin trees. Therefore, CBP is stronger than the Projective Stationary Reflection Principle.

A restatement of Theorem 2 of Todorcevic [18] shows that if (*) every stationary set $S \subseteq [\omega_2]^\omega$ reflects (i.e., there is some $\delta < \omega_2$ such that $S \cap [\delta]^\omega$ is stationary in $[\delta]^\omega$), then $\mathbb{R}^M = \mathbb{R}$ for every inner model M satisfying that $(\omega_2)^M = \omega_2$. (To see this, let M be an inner model such that $\aleph_2^M = \alpha_2$. Let $\langle e_\alpha | \alpha < \omega_2 \rangle$ be in M such that $e_\alpha \colon \alpha \to |\alpha|$ be a bijection for $\alpha < \omega_2$. Let

$$C = \{x \in [\omega_2]^\omega | \forall \alpha \in x e''_\alpha x \subseteq x, \& (e_\alpha^{-1})'' x \subseteq x\}.$$

Consider

$$S = \{x \in C | \forall \alpha < \omega_2 \forall \beta < \omega_1 x \neq e''_\alpha \beta\}.$$

CLAIM: *If $\mathbb{R}^M \neq \mathbb{R}$, then S is projective stationary in $[\omega_2]^\omega$.)*

This follows from a theorem of Gitik–Velickovic that $[\omega_2]^\omega - M$ is projective stationary in $[\omega_2]^\omega$ (see [8] Theorem 1.1 and [21], Lemma 3.15).

Notice that this set S never reflects.

It then follows that when a real is added to a model of (*) while preserving $\aleph_1$ and $\aleph_2$, then (*) must fail. In particular, the Semiproper Forcing Axiom, CBP, Strong Reflection Principle, Reflection Principle and Strong Chang's Conjecture are all forced to be false when a Cohen real is added to a model in which they were true. Similarly, Rado's Conjecture is forced to be false when a Cohen real is added to a model in which it was true. This shows that all these strong properties are easily destroyed. However, another type of reflection is not so easily destroyed.

FACT: Let $\kappa \geq \omega_2$ be regular. If every stationary $S \subseteq \{\alpha < \kappa | cf(\alpha) = \omega\}$ reflects, if P is the forcing adding one Cohen real, then it is forced that every stationary $S \subseteq \{\alpha < \kappa | cf(\alpha) = \omega\}$ reflects.

To see this, let $\dot{E}$ be a name such that

$$p \Vdash \dot{E} \subseteq \{\alpha < \kappa | cf(\alpha) = \omega\} \text{ is stationary in } \kappa.$$

Let $\langle p_n | n < \omega \rangle$ be an enumeration of all conditions below p. For each $n < \omega$, let

$$E_n = \{\alpha < \kappa | p_n \Vdash \alpha \in \dot{E}\}.$$

Then there is some $n < \omega$ such that E_n is stationary. Hence E_n reflects to some $\delta < \kappa$. Then p_n forces that $\dot{E} \cap \delta$ is stationary in δ.

In [1], Abraham and Shelah essentially prove that (see [1] Theorem 6) if M is an inner model such that $\aleph_2^M = \aleph_2$, then $S = ([\omega_2]^\omega)^M$ is projective stationary in $[\omega_2]^\omega$.

(To see this, let $f: [\omega_2]^{<\omega} \to \omega_2$. Let $T \subseteq \omega_1$ be stationary. Let $\omega_1 < \alpha < \omega_2$ be such that α is closed under f. Let $h \in M$ be a bijection of α to $\aleph_1$. The set

$$C = \{h''\beta |\ \beta < \omega_1\}$$

is in M and is a club in $[\alpha]^\omega$ in the real world. Let $\gamma \in T$ be such that $\gamma = \omega_1 \cap h''\gamma$ and $h''\gamma$ is closed under f. Then $h''\gamma \in S$.

Hence S is projective stationary.) ∎

References

[1] U. Abraham and S. Shelah, *Forcing closed unbounded sets,* Journal of Symbolic Logic **48** (1983), 643–657.

[2] J. Baumgartner, *Applications of the Proper Forcing Axiom,* in *Handbook of Set-Theoretic Topology* (K. Kunen and J. E. Vaughan, eds.), Elsevier Science Publishers B.V., Amsterdam, 1984, pp. 913–959.

[3] M. Bekkali, *Topics in Set Theory,* Lecture Notes in Mathematics, Vol. 1476, Springer-Verlag, Berlin, New York, 1991.

[4] Q. Feng, *Rado's conjecture and presaturation of the nonstationary ideal on* ω_1, Journal of Symbolic Logic **64** (1999), 38–44.

[5] Q. Feng and T. Jech, *Local clubs, reflection, and preserving stationary sets,* Proceedings of the London Mathematical Society (3) **58** (1989), 237–257.

[6] Q. Feng and T. Jech, *Projective stationary sets and strong reflection principles,* Journal of the London Mathematical Society (2) **58** (1998), 271–283.

[7] M. Foreman, M. Magidor and S. Shelah, *Martin's maximum, saturated ideals, and nonregular ultrafilters. Part I,* Annals of Mathematics **127** (1988), 1–47.

[8] M. Gitik, *Nonsplitting subset of* $P_\kappa(\kappa^+)$, Journal of Symbolic Logic **50** (1985), 881–894.

[9] J. Gregory, *A countable distributive complete Boolean algebra not uncountably representable,* Proceedings of the American Mathematical Society **42** (1974), 42–46.

[10] T. Jech, *Some combinatorial problems concerning uncountable cardinals,* Annals of Mathematical Logic **5** (1973), 165–198.

[11] T. Jech, *Set Theory*, Academic Press, New York, 1978.

[12] T. Jech, *More game-theoretic properties of boolean algebras*, Annals of Pure and Applied Logic **26** (1984), 11–29.

[13] T. Jech, *Multiple Forcing*, Cambridge Tracts in Mathematics, Cambridge University Press, 1986.

[14] D. Kueker, *Countable approximations and Löwenheim–Skolem theorem*, Annals of Mathematical Logic **11** (1977), 57–103.

[15] S. Shelah, *Semiproper Forcing Axiom implies Martin Maximum but not PFA$^+$*, Journal of Symbolic Logic **52** (1987), 360–367.

[16] S. Todorcevic, *On a conjecture of R. Rado*, Journal of the London Mathematical Society **2** (1983), 1–8.

[17] S. Todorcevic, *Partition relations for partially ordered sets*, Acta Mathematica **155** (1985), 1–25.

[18] S. Todorcevic, *Reflecting Stationary Sets I*, Handwritten Notes, 1985.

[19] S. Todorcevic, *Strong Reflections*, Handwritten Notes, 1987.

[20] S. Todorcevic, *Conjectures of Rado and Chang and Cardinal Arithmetic*, in *Finite and Infinite Combinatorics in Sets and Logic* (Banff, AB, 1991), NATO Advanced Science Institutes Series C: Mathematical and Physical Sciences, 441, Kluwer Academic, Dordrecht, 1993, pp. 385–398.

[21] B. Velickovic, *Forcing axioms and stationary sets*, Advances in Mathematics **94** (1992), 256–284.

[22] H. Woodin, Lecture Notes, 1993.